3.5 Graphing Sine and Cosine Functions

Any equation of the form $y = a\sin\left[b(x-c)\right]+d$ with $a \neq 0$ and $b \neq 0$ is a **sine function**. Its graph is called a **sine wave**, **sinusoidal wave**, or **sinusoid.** The graph of any sine function is a transformation of the graph of $y = \sin x$.

We assume x is in radians unless the problem specifically states that it is in degrees.

As the terminal side of an angle rotates around the unit circle, how does the value of the sine change?

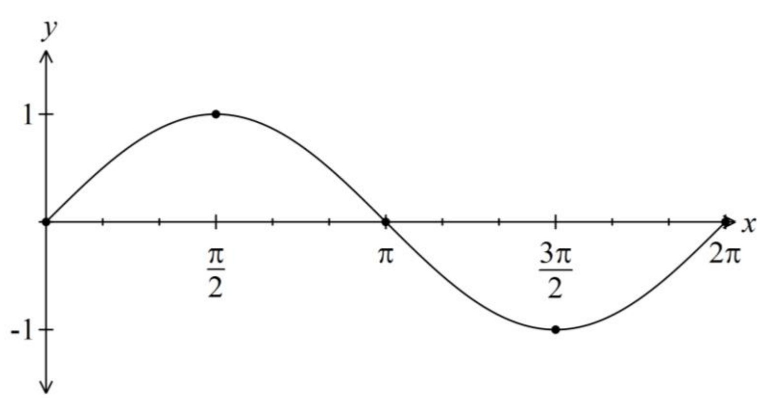
- From 0 to $\pi/2$, the sine increases from 0 to 1.
- From $\pi/2$ to π, the sine decreases from 1 to 0.
- From π to $3\pi/2$, the sine decreases from 0 to –1.
- From $3\pi/2$ to 2π, the sine increases from –1 to 0.
- The cycle repeats.

Because $\sin(x+2\pi) = \sin x$, the shape we see in the interval $[0, 2\pi]$ repeats on the intervals $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, $[-2\pi, 0]$, $[-4\pi, -2\pi]$, etc.

A repeating function like $y = \sin x$ is called a **periodic function.** The length of the smallest non-repeating unit is the **period** of the function. The period of $y = \sin x$ is 2π. The graph of $y = \sin x$ over any interval of length 2π is called a **cycle.** The graph of $y = \sin x$ over $[0, 2\pi]$ is the **fundamental cycle.**

Key points on the graph of $y = \sin x$:

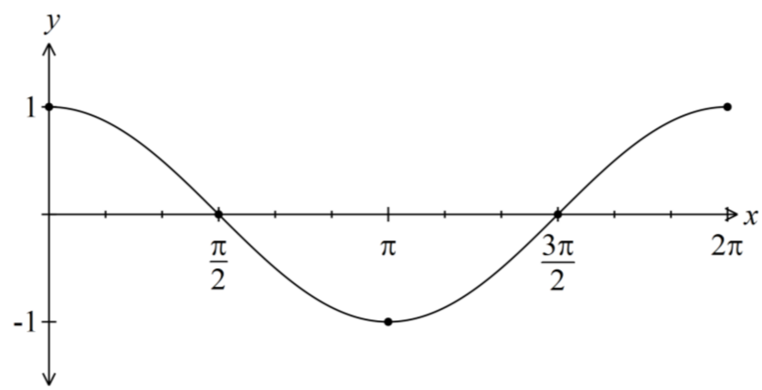
x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \sin x$	0	1	0	–1	0



The graph of $y = \cos x$ has the same shape as the graph of $y = \sin x$, but it is shifted to the left by a distance of $\pi/2$. For this reason, the graph of $y = \cos x$ is also called a sine wave. The graph of $y = \cos x$ over $[0, 2\pi]$ is called the **fundamental cycle** of $y = \cos x$.

Key points on the graph of $y = \cos x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \cos x$	1	0	-1	0	1

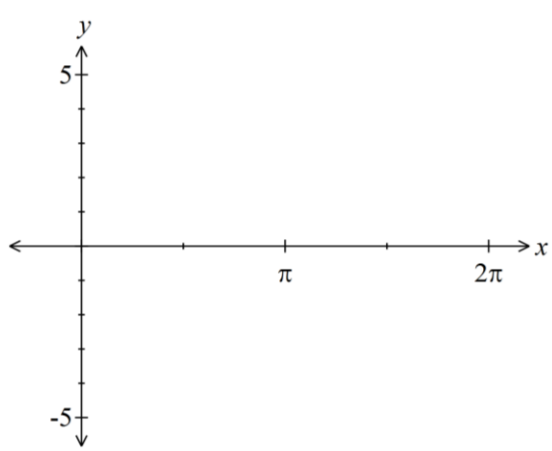


The effect of changing the value of *a*:

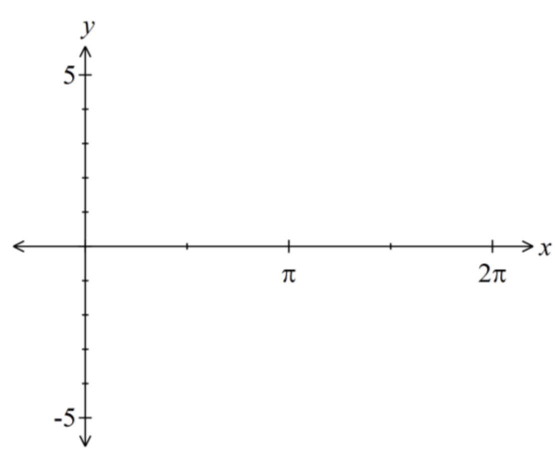
The **amplitude** of $y = a\sin x$ or $y = a\cos x$ is $|a|$. The amplitude is the "height" of the sine wave. It is half the difference between the maximum and minimum points on the graph. If a is negative, the graph is reflected over the x-axis.

Examples: Sketch the graphs of the following and determine the amplitude and range of each.

$y = 3\sin x$



$y = -5\cos x$

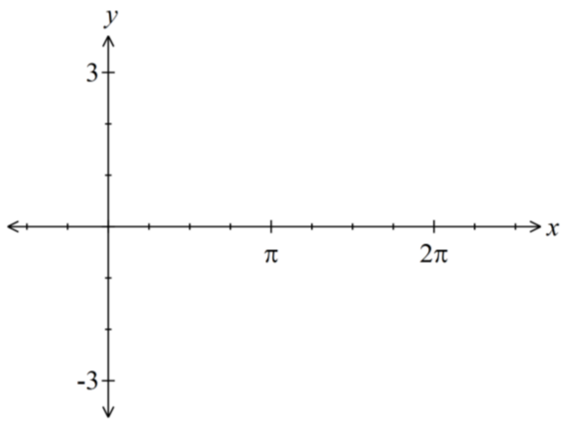


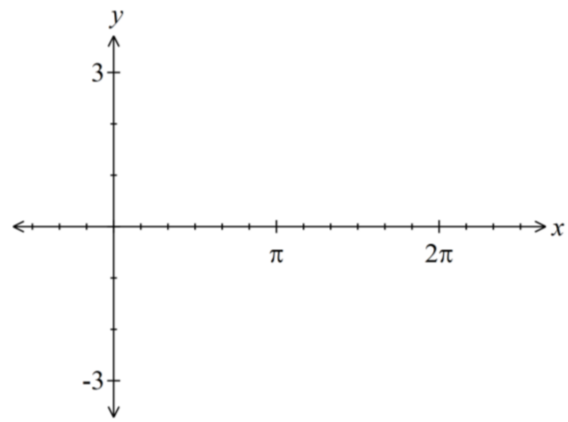
The effect of changing the value of *c*:

The **phase shift** of the graph of $y = \sin(x - c)$ or $y = \cos(x - c)$ is c. Notice that the sign of *c* is the opposite of the sign in the equation. This means that the graph is shifted *c* units to the right if *c* is positive, or *c* units to the left if *c* is negative.

Examples: Sketch each graph and find the amplitude, phase shift, and range of each function.

$y = \cos(x - \pi/4)$ $\qquad\qquad$ $y = 2\sin(x + \pi/3)$



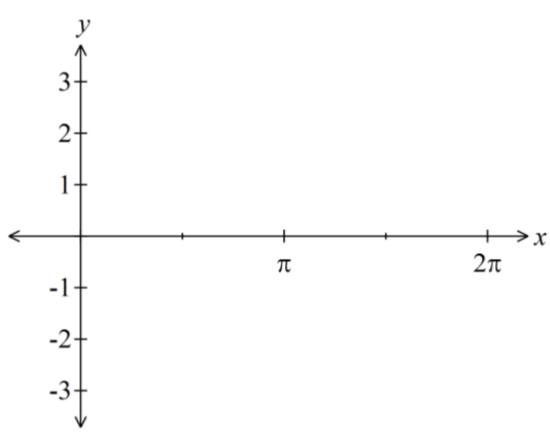


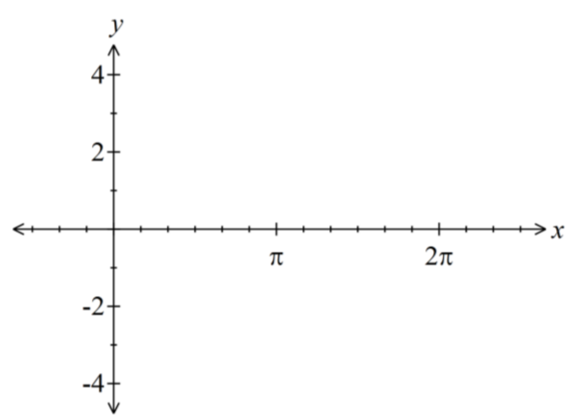
The effect of changing the value of *d*:

The **vertical translation** of the graph of $y = \sin x + d$ or $y = \cos x + d$ is d. This means that the graph is shifted *d* units up if *d* is positive, or *d* units down if *d* is negative.

Examples: Sketch each graph and find the amplitude, phase shift, vertical shift, and range of each function

$y = \sin x + 2$ $\qquad\qquad$ $y = 3\cos(x - \pi/6) - 1$





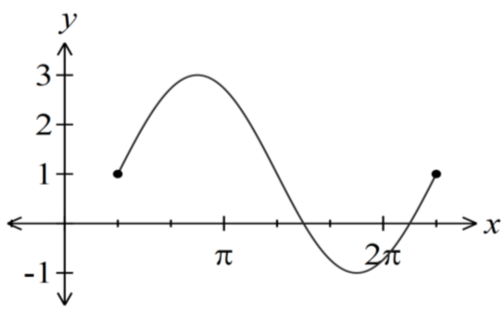
Examples: Find the equation of each sine wave in its final position.

1. The graph of $y = \sin x$ is stretched by a factor of 2, reflected in the *x*-axis, shifted $\pi/5$ units to the right, then translated 4 units downward.

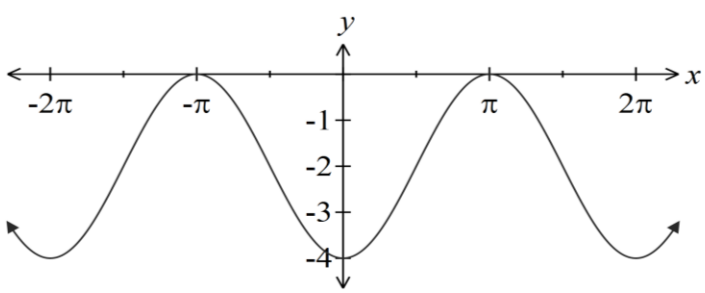
2. The graph of $y = \cos x$ is shifted $\pi/3$ units to the left, translated upward 2 units, then stretched by a factor of 2.

Examples: Find an equation of the requested form whose graph is the given sine wave.

$y = a\sin(x - c) + d$



$y = a\cos(x - c) + d$

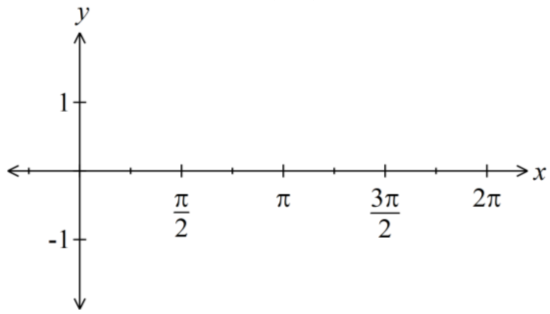


The effect of changing the value of *b*:

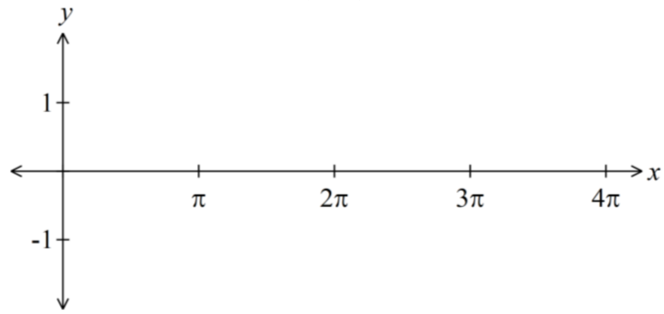
The **period** of the graph of $y = \sin(bx)$ or $y = \cos(bx)$ for $b > 0$ is $P = 2\pi/b$. This means that there are b cycles every 2π units. The **frequency**, F, of a sine wave with period P is defined by $F = 1/P = b/2\pi$.

Examples: Sketch the graphs of the following and determine the period and frequency of each.

$y = \sin(2x)$



$y = \cos(x/2)$



The general sine wave:

Characteristics of the graph of $y = a\sin[b(x-c)] + d$ or $y = a\cos[b(x-c)] + d$:

- Amplitude: $|a|$
- Period: $P = 2\pi/b$
- Frequency: $F = 1/P = b/2\pi$
- Phase shift: c (Remember that the sign of c is the opposite of the sign in the equation).
    - Shift right for $c > 0$.
    - Shift left for $c < 0$.
- Vertical translation: d
    - Shift up for $d > 0$.
    - Shift down for $d < 0$.

Steps to graph $\boldsymbol{y = a\sin[b(x-c)] + d}$ or $\boldsymbol{y = a\cos[b(x-c)] + d}$:

Start with the five key points on the graph of $y = \sin x$ or $y = \cos x$.

1. Find five key points for $y = a\sin[b(x-c)] + d$ or $y = a\cos[b(x-c)] + d$ by
    a. dividing each x-coordinate by b and adding c.
    b. multiplying each y-coordinate by a and adding d.
    c. sketch one cycle of your graph through the five new points.

***Note:** Order is important. Multiply or divide first, then add.

Examples: Determine the amplitude, period, frequency, phase shift, and vertical shift of the following. Then sketch one cycle of each graph. Draw and label your own axes.

$$y = \sin\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] + 1$$

$$y = 2\cos\left(2x + \frac{\pi}{2}\right) - 2$$

$$y = -3\cos\left[3\left(x - \frac{2\pi}{3}\right)\right] - 1$$

$$y = 4\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right)$$